Int. J. Heat Mass Transfer. Vol. 34, No. 11, pp. 2911–2919, 1991
Printed in Great Britain

0017-9310/91 \$3.00+0.00

A general optimization method using adjoint equation for solving multidimensional inverse heat conduction

Y. JARNY, M. N. OZISIK† and J. P. BARDON
University of Nantes, I.S.I.T.E.M., La Chantrerie CP3023, 44087 Nantes Cedex 03, France

(*Received* 31 *October* 1990)

Abstract—A three-dimensional formulation is presented to solve inverse heat conduction as a general optimization problem by applying the adjoint equation approach coupled to the conjugate gradient algorithm. The formulation consists of the sensitivity problem, the adjoint problem and the gradient equations. A solution algorithm is presented for the estimation of the surface condition (i.e. heat flux or temperature), space dependent thermal conductivity and heat capacity from the knowledge of transient temperature recordings taken within the solid. In this approach, no a priori information is needed about the unknown function to be determined. It is shown that the problems involving a priori information about the unknown function become special cases of this general approach.

1. INTRODUCTION

THE USE of inverse analysis for the estimation of surface conditions such as temperature and heat flux, or the determination of thermal properties such as thermal conductivity and heat capacity of solids by utilizing the transient temperature measurements taken within the medium, has numerous practical applications. For example, the direct measurement of heat flux at the surface of a wall subjected to fire, at the outer surface of a re-entry vehicle or at the inside surface of a combustion chamber is extremely difficult. In such situations, the inverse method of analysis, using transient temperature measurements taken within the medium can be applied for the estimation of such quantities. However, difficulties associated with the implementation of inverse analysis should also be recognized. The main difficulty comes from the fact that inverse problems are ill-posed, the solutions are very sensitive to changes in input data resulting from measurement and modelling errors, hence may not be unique. An excellent discussion of difficulties encountered in inverse analysis is well documented in the text on inverse heat conduction [1]. To overcome such difficulties a variety of techniques for solving inverse heat conduction problems have been proposed in the literature [1–7]. The use of the adjoint equation approach coupled to the conjugate gradient [8–13] appears to be very powerful for solving inverse heat conduction problems.

The mathematical formulation of this method consists of the development of the sensitivity problem, the adjoint problem and the gradient equations. The type of boundary conditions as well as the nature of the inverse problem affect the formulation. Therefore, the objective of this work is to present a multidimensional unified formulation of the adjoint equation approach for solving inverse heat conduction problems for situations in which no a priori information is available about the unknown function.

In Section 2, the inverse problem is formulated as an optimization problem over a space function and in Section 3 the sensitivity problem is introduced. In Section 4, the adjoint problem and the gradient equations are developed and in Section 5, it is shown that the finite dimensional situation, that is, the problem with a priori information about the function, becomes a special case of the present method. Finally, in Section 6, an algorithm is presented for the solution of inverse transient heat conduction by the conjugate gradient method.

2. FORMULATION OF THE INVERSE PROBLEM

2.1. *The direct problem*

We consider the following three-dimensional, linear, direct, transient heat conduction problem in a region $\mathscr{R}$, over the time interval from the initial time $t = 0$ to the final time $t = t_f$

$$C(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial t} - \nabla \cdot \lambda(\mathbf{r})\nabla T(\mathbf{r}, t) = g(\mathbf{r}, t), \quad \text{in } \mathscr{R}. \tag{1a}$$

In order to illustrate the implications of different types of boundary conditions in the formulation of the inverse problem, we consider three different linear boundary conditions, namely, convection, prescribed heat flux and prescribed temperature on three different boundary surfaces A_1, A_2 and A_3, respectively

† Permanent address: MAE Department, Box 7910, North Carolina State University, Raleigh, NC 27695, U.S.A.

NOMENCLATURE

A_i	boundary surface i	t_f	final time
$C(\mathbf{r})$	heat capacity	T	temperature
$D_{\Delta Z}J(Z)$	directional derivative of J at Z	$Y_m(t)$	measurement function, defined by equation (15)
d_m	sensor location in the medium	Z	unknown function
E	function space in which Z is to be found	Z_{est}	estimated value of Z.
$e_m(t)$	error term defined by equation (16a)		
$f_1(\mathbf{r}, t)$	boundary condition function for surface A_1, equation (1b)	Greek symbols	
$f_2(\mathbf{r}, t)$	boundary condition function for surface A_2, equation (1c)	α_0	positive regularization parameter
$G(\mathbf{r})$	defined by equation (29c)	β^n	defined by equation (40b)
$h(\mathbf{r})$	heat transfer coefficient	γ_i	defined by equation (33)
$H(t)$	defined by equation (27b)	γ_i^*	defined by equation (38)
$J(Z)$	functional defined by equation (4)	ε	real number
K^n	defined by equation (43b)	$\theta(\mathbf{r}, t)$	sensitivity function defined by problem, equations (15)
M	total number of measurement locations	$\lambda(\mathbf{r})$	thermal conductivity
NT	total number of time measurements	ρ	step size defined by equation (40e)
p^n	direction of descent, defined by equation (40d)	σ_i	basis function defined by equation (7)
S_{ij}	defined by equation (34)	$\psi(\mathbf{r}, t)$	adjoint function defined by problem, equations (16b)–(16f).

$$\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_1} + h(\mathbf{r})T(\mathbf{r}, t) = f_1(\mathbf{r}, t), \quad \text{on } A_1 \tag{1b}$$

$$\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_2} = f_2(\mathbf{r}, t), \quad \text{on } A_2 \tag{1c}$$

$$T(\mathbf{r}, t) = T_3(\mathbf{r}, t), \quad \text{on } A_3 \tag{1d}$$

$$T(\mathbf{r}, 0) = T_0(\mathbf{r}), \quad \text{in } \mathscr{R} \tag{1e}$$

where $C(\mathbf{r})$, $\lambda(\mathbf{r})$ and $h(\mathbf{r})$ are strictly positive, A_i, $i = 1$, 2 and 3 are continuous boundary surfaces of the region $\mathscr{R}$, and $\partial/\partial n_i$ the derivative along the outward-drawn normal to the boundary surfaces A_i, $i = 1, 2$.

The physical significance of the function $f_1(\mathbf{r}, t)$ in equation (1b) is

$$f_1(\mathbf{r}, t) \equiv h(\mathbf{r})\phi(\mathbf{r}, t), \quad \text{for } h(\mathbf{r}) \neq 0$$

where $\phi(\mathbf{r}, t)$ is the ambient temperature. As a special case of this general formulation, the following form is of practical interest:

$$f_1(\mathbf{r}, t) \equiv h(\mathbf{r})\phi(t) \tag{1f}$$

where the ambient temperature $\phi(t)$ varies with time only.

2.2. *The measured temperature data*

We assume that there are M sensors located at the positions

$$\mathbf{r} = d_m, \quad d_m \in \mathscr{R} \tag{2a}$$

for which temperature observations

$$Y_m(t_k) \equiv Y_m^k \tag{2b}$$

are available at times t_k, $0 < t_k < t_f$, $k = 1, \ldots, NT$ and positions d_m, $m = 1, \ldots, M$.

2.3. *The inverse problem*

We denote, by Z, the function to be determined by the inverse analysis and Z_{est} its estimated value, if available. In the present problem, the function Z can be any one of the following quantities:

$Z_1(\mathbf{r}, t) \equiv f_1(\mathbf{r}, t)$; where $f_1(\mathbf{r}, t)$ at the boundary surface A_1 is related to the ambient temperature $\phi(\mathbf{r}, t)$ through the relation $f_1(\mathbf{r}, t) \equiv h(\mathbf{r})\phi(\mathbf{r}, t)$

$Z_2(\mathbf{r}, t) \equiv f_2(\mathbf{r}, t)$ the surface heat flux of A_2

$Z_3(\mathbf{r}, t) \equiv T_3(\mathbf{r}, t)$ the surface temperature on A_3

$Z_4(\mathbf{r}) \equiv \lambda(\mathbf{r})$ the thermal conductivity

$Z_5(\mathbf{r}) \equiv C(\mathbf{r})$ the heat capacity

$Z_6(t) \equiv \phi(t)$ the ambient temperature, independent of position.

Let $T(\mathbf{r}, t; Z)$ denote the solution of the direct problem; that is, the temperature corresponding to a particular value of the unknown function Z.

The inverse problem for the ideal situation is defined as follows:

Find Z such that

$$Y_m^k = T(d_m, t_k; Z) \tag{3}$$

for $k = 1$ to NT and $m = 1$ to M.

Because of measurement or model errors, this equation needs to be solved in the least square sense. Then, the inverse problem is defined as follows:

Find $Z \in E$ which minimizes the functional $J(Z)$ defined by

$$J(Z) = \frac{1}{2}\int_0^{t_f} \sum_{m=1}^{M} |T(d_m, t; Z) - \bar{Y}_m(t)|^2 \, dt + \frac{1}{2}\alpha_0 \|Z - Z_{est}\|_E^2 \quad (4)$$

where $\bar{Y}_m(t)$ is the measurement function considered constant over the time interval $[t_k, t_{k+1}[$ and defined by

$$\bar{Y}_m(t) = Y_m^k, \quad t \in [t_m, t_{k+1}[; \quad k = 1 \text{ to } NT \quad (5)$$

α_0 is a positive regularization parameter, E the function space in which Z is to be found, and $\| \ \|_E^2$ the norm associated to the scalar product $\langle .,. \rangle_E$.

In the present problem, the following three possibilities are considered for E:

(i) $E = L^2(0, t_f)$ = space of square integrable functions on $]0, t_f[$, with the scalar product defined by

$$\langle Z_6, \bar{Z}_6 \rangle_E = \int_0^{t_f} Z_6(t)\bar{Z}_6(t)\, dt. \quad (6a)$$

(ii) $E = L^2(\mathscr{R})$ = all square integrable functions on $\mathscr{R}$, with the scalar product defined by

$$\langle Z_i, \bar{Z}_i \rangle_E = \int_{\mathscr{R}} Z_i(\mathbf{r})\bar{Z}_i(\mathbf{r})\, d\mathbf{r} \quad i = 4 \text{ or } 5. \quad (6b)$$

(iii) $E = L^2(A_i]0, t_f[)$ = all square integrable functions defined on $A_1 \times]0, t_f[$, with the scalar product

$$\langle Z_i, \bar{Z}_i \rangle_E = \int_0^{t_f} \int_{A_i} Z_i(\mathbf{r}, t)\bar{Z}_i(\mathbf{r}, t)\, d\mathbf{r}\, dt \quad i = 1, 2 \text{ or } 3. \quad (6c)$$

More regular function spaces can be chosen, for example, with square integrable first derivative functions [14].

The dimensionality of this optimization problem is characterized by the dimension of the function space E over which the minimization occurs.

The functional optimization approach considered here does not require any a priori information on the nature of the function to be determined, hence, in general it is *infinite dimensional.* For the special case, when a priori information is available on the nature of the unknown function $Z(s)$, then $Z(s)$ may be represented in the form

$$Z(s) = \sum_{i=1}^{P} Z_i \sigma_i(s) \quad (7)$$

where

$\{\sigma_i, i = 1 \text{ to } P\}$ is a set of basis functions of E

$\{Z_i, i = 1 \text{ to } P\}$ is a P-dimensional vector in $\mathbf{R}^P$.

For such a case, the optimization problem becomes *finite dimensional* and the standard least square method as well as the present method can be used for the solution.

For the finite dimensional problem, the gradient $\nabla J(Z)$ is readily determined by the standard differential calculus; but for the infinite dimensional problem, it is necessary to develop an adjoint problem in order to compute the gradient $\nabla J(Z)$ needed in the minimization process. The use of the present method for the solution of finite dimensional problems will be described in Section 5.

2.4. *Definition of* $\nabla J(Z)$

The *gradient* of the functional $J(Z)$, at point Z, denoted by $\nabla J(Z)$, is related to the variation of J at this point by the general equation

$$J(Z + \varepsilon\Delta Z) - J(Z) = \langle \nabla J(Z), \varepsilon\Delta Z \rangle_E + (\text{terms nonlinear in } \|\Delta Z\|) \quad (8)$$

where $(Z + \varepsilon\Delta Z) \in E$ and ε = real number.

The *directional derivative* of J at Z in the direction ΔZ, denoted by $D_{\Delta Z}J(Z)$, is defined by

$$D_{\Delta Z}J(Z) = \lim_{\varepsilon \to 0} \frac{J(Z + \varepsilon\Delta Z) - J(Z)}{\varepsilon} \quad (9a)$$

and it is related to the gradient $\nabla J(Z)$, by

$$D_{\Delta Z}J(Z) = \langle \nabla J(Z), \Delta Z \rangle_E. \quad (9b)$$

For example, if $E = L^2(0, t_f)$, then according to equation (6a), $D_{\Delta Z}J(Z)$ becomes

$$D_{\Delta Z}J(Z) = \int_0^{t_f} \nabla J(t; Z)\Delta Z(t)\, dt. \quad (10)$$

In the following sections, by introducing sensitivity and adjoint problems, we develop explicit expressions for $\nabla J(Z)$ in the form given by equations (9).

3. THE SENSITIVITY PROBLEM

3.1. *Definitions*

Let $\Delta T_{\varepsilon\Delta Z}$ be the increment of temperature resulting from the change of the unknown function Z in the amount $\varepsilon\Delta Z$, that is

$$\Delta T_{\varepsilon\Delta Z} = T(\mathbf{r}, t; Z + \varepsilon\Delta Z) - T(\mathbf{r}, t; Z). \quad (11)$$

The directional derivative of T, $D_{\Delta Z}T(Z)$, evaluated at $(\mathbf{r}, t)$ in the direction ΔZ, is defined in the same way as in equations (9), that is

$$D_{\Delta Z}T(\mathbf{r}, t; Z) = \lim_{\varepsilon \to 0} \frac{T(\mathbf{r}, t; Z + \varepsilon\Delta Z) - T(\mathbf{r}, t; Z)}{\varepsilon}. \quad (12a)$$

We note that $D_{\Delta Z}T(\mathbf{r}, t; Z)$ is a sensitivity function which will be denoted by $\theta(\mathbf{r}, t)$; that is

$$D_{\Delta Z}T(\mathbf{r}, t; Z) \equiv \theta(\mathbf{r}, t). \quad (12b)$$

To determine the problem defining the sensitivity function, the direct problem given by system (1) is written first for $(Z + \varepsilon\Delta Z)$, then for Z, the results are subtracted, and the limiting process defined by equation (12a) is applied.

To illustrate the procedure, we consider the following two specific examples.

Case #1. The function $f_1(\mathbf{r}, t)$ unknown. The function $f_1(\mathbf{r}, t)$ is unknown over the boundary surface A_1 and we wish to determine it by the inverse analysis. As discussed previously, $f_1(\mathbf{r}, t)$ is related to the ambient temperature $\phi(\mathbf{r}, t)$ through the relation $f_1(\mathbf{r}, t) = h(\mathbf{r})\phi(\mathbf{r}, t)$. Equations (1a)–(1e) being linear, the sensitivity problem for this case is immediately written as

$$C(\mathbf{r})\frac{\partial\theta(\mathbf{r}, t)}{\partial t} - \nabla\cdot\lambda(\mathbf{r})\nabla\theta(\mathbf{r}, t) = 0, \quad \text{in } \mathscr{R} \quad (13a)$$

$$\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r}, t)}{\partial n_1} + h(\mathbf{r})\theta(\mathbf{r}, t) = \Delta f_1(\mathbf{r}, t), \quad \text{on } A_1 \quad (13b)$$

$$\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r}, t)}{\partial n_2} = 0, \quad \text{on } A_2 \quad (13c)$$

$$\theta(\mathbf{r}, t) = 0, \quad \text{on } A_3 \quad (13d)$$

$$\theta(\mathbf{r}, 0) = 0, \quad \text{in } \mathscr{R}. \quad (13e)$$

Case #2. The thermal conductivity unknown. The thermal conductivity $Z = \lambda(\mathbf{r})$ in the medium is to be determined. For this case, by setting $T(\mathbf{r}, t; Z + \varepsilon\Delta Z) \equiv T(\mathbf{r}, t; \lambda + \varepsilon\Delta\lambda)$ in equations (1a)–(1c), system (1) becomes

$$C(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial t} - \nabla\cdot\lambda(\mathbf{r})\nabla T(\mathbf{r}, t) = g(\mathbf{r}, t) + \nabla\varepsilon\Delta\lambda(\mathbf{r})\nabla T(\mathbf{r}, t), \quad \text{in } \mathscr{R} \quad (14a)$$

$$\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_1} + h(\mathbf{r})T(\mathbf{r}, t) = f_1(\mathbf{r}, t) - \varepsilon\Delta\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_1}, \quad \text{on } A_1 \quad (14b)$$

$$\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_2} = f_2(\mathbf{r}, t) - \varepsilon\Delta\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_2}, \quad \text{on } A_2 \quad (14c)$$

$$T(\mathbf{r}, t) = T_3(\mathbf{r}, t), \quad \text{on } A_3 \quad (14d)$$

$$T(\mathbf{r}, 0) = T_0(\mathbf{r}), \quad \text{in } \mathscr{R}. \quad (14e)$$

Then by applying the definition of the direction derivative defined by equation (12a), the sensitivity problem for this case, takes the form

$$C(\mathbf{r})\frac{\partial\theta(\mathbf{r}, t)}{\partial t} - \nabla\cdot\lambda(\mathbf{r})\nabla\theta(\mathbf{r}, t) = \nabla\Delta\lambda(\mathbf{r})\nabla T(\mathbf{r}, t), \quad \text{in } \mathscr{R} \quad (15a)$$

$$\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r}, t)}{\partial n_1} + h(\mathbf{r})\theta(\mathbf{r}, t) = -\Delta\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_1}, \quad \text{on } A_1 \quad (15b)$$

$$\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r}, t)}{\partial n_2} = -\Delta\lambda(\mathbf{r})\frac{\partial T(\mathbf{r}, t)}{\partial n_2}, \quad \text{on } A_2 \quad (15c)$$

$$\theta(\mathbf{r}, t) = 0, \quad \text{on } A_3 \quad (15d)$$

$$\theta(\mathbf{r}, 0) = 0, \quad \text{in } \mathscr{R}. \quad (15e)$$

Depending on the nature of the inverse problem, the sensitivity function $\theta(\mathbf{r}, t)$ either depends on the temperature $T(\mathbf{r}, t)$ or is independent of it. In the above examples, $\theta(\mathbf{r}, t)$ depends on $T(\mathbf{r}, t)$ in case #2 and is independent of it in case #1.

4. THE ADJOINT PROBLEM AND THE GRADIENT EQUATIONS

As discussed previously, for infinite dimensional problems, an adjoint function $\psi(\mathbf{r}, t)$ is needed to determine the gradient $\nabla J(Z)$ of the functional in addition to the sensitivity function, $\theta(\mathbf{r}, t)$.

We develop below first the adjoint problem and then the gradient equations.

4.1. *Adjoint problem*

We consider the error terms $e_m(t)$ defined by

$$e_m(t) = T(d_m, t; Z) - \bar{Y}_m(t), \quad m = 1, \ldots, M. \quad (16a)$$

Then the adjoint function $\psi(\mathbf{r}, t)$ is taken as the solution of the following linear problem:

$$-C(\mathbf{r})\frac{\partial\psi(\mathbf{r}, t)}{\partial t} - \nabla(\lambda(\mathbf{r})\nabla\psi(\mathbf{r}, t)) = \sum_{m=1}^{M} e_m(t)\cdot\delta(\mathbf{r} - d_m), \quad \text{in } \mathscr{R} \quad (16b)$$

$$\lambda(\mathbf{r})\frac{\partial\psi(\mathbf{r}, t)}{\partial n_1} + h(\mathbf{r})\psi(\mathbf{r}, t) = 0, \quad \text{on } A_1 \quad (16c)$$

$$\lambda(\mathbf{r})\frac{\partial\psi(\mathbf{r}, t)}{\partial n_2} = 0, \quad \text{on } A_2 \quad (16d)$$

$$\psi(\mathbf{r}, t) = 0, \quad \text{on } A_3 \quad (16e)$$

$$\psi(\mathbf{r}, t_f) = 0, \quad \text{in } \mathscr{R} \quad (16f)$$

where $\delta(\cdot)$ is the Dirac delta function.

Clearly, with no error, the adjoint problem has zero solution. We note that in the adjoint problem, the time is measured backwards from the final time t_f to the initial time $t = 0$. However, by defining a new time variable $\tau = t_f - t$, the corresponding τ domain becomes from $\tau = 0$ to t_f.

4.2. *Gradient equations*

We start with the definition of the functional $J(Z)$ given by equation (4), and compute the directional derivative $D_{\Delta Z}J(Z)$ of J at Z in the direction ΔZ, according to definitions (9) and (12) and obtain

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\sum_{m=1}^{M}(T(d_m, t; Z) - \bar{Y}_m(t)) \times (D_{\Delta Z}T(d_m, t; Z))\,\mathrm{d}t + \alpha_0\langle Z - Z_{\text{est}}, \Delta Z\rangle_E. \quad (17)$$

The integral term appearing on the right-hand side of equation (17) is written in the form

$$\int_0^{t_f}\sum_{m=1}^{M}(T(d_m, t; Z) - \bar{Y}_m(t))(D_{\Delta Z}T(d_m, t; Z))\,\mathrm{d}t = \int_0^{t_f}\int_{\mathscr{R}}\sum_{m=1}^{M}(T(d_m, t; Z) - \bar{Y}_m(t)) \times \delta(\mathbf{r} - d_m)\theta(\mathbf{r}, t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t \quad (18)$$

where we utilized the definition of the delta function at $\mathbf{r} = d_m$, and the sensitivity function.

Then equation (17) takes the form

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\int_{\mathscr{R}}\sum_{m=1}^{M}(T(d_m,t;Z)-\tilde{Y}_m(t))\times\delta(\mathbf{r}-d_m)\theta(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (19a)$$

Equations (16a) and (16b) are now utilized to write equation (19a) as

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\int_{\mathscr{R}}\left(-C(\mathbf{r})\frac{\partial\psi(\mathbf{r},t)}{\partial t}-\nabla(\lambda(\mathbf{r})\nabla\psi(\mathbf{r},t))\right)\times\theta(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (19b)$$

Integrating by parts with respect to t and using Green's formula, equation (19b) takes the form

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\int_{\mathscr{R}}\left(C(\mathbf{r})\frac{\partial\theta(\mathbf{r},t)}{\partial t}-\nabla\lambda(\mathbf{r})\nabla\theta(\mathbf{r},t)\right)\times\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t+\sum_i\int_0^{t_f}\int_{A_i}\left(\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r},t)}{\partial n_i}\psi(\mathbf{r},t)-\lambda(\mathbf{r})\theta(\mathbf{r},t)\frac{\partial\psi(\mathbf{r},t)}{\partial n_i}\right)\mathrm{d}A\,\mathrm{d}t-\int_{\mathscr{R}}(C(\mathbf{r})\psi(\mathbf{r},t_f)\theta(\mathbf{r},t_f)-C(\mathbf{r})\psi(\mathbf{r},0)\theta(\mathbf{r},0))\,\mathrm{d}r+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (20)$$

Applying the boundary conditions given by equations (16c)–(16e) and the final condition given by equation (16f), this result is written as

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\int_{\mathscr{R}}\left(C(\mathbf{r})\frac{\partial\theta(\mathbf{r},t)}{\partial t}-\nabla\cdot\lambda(\mathbf{r})\nabla\theta(\mathbf{r},t)\right)\times\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t+\int_0^{t_f}\int_{A_1}\left(\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r},t)}{\partial n_1}+h(\mathbf{r})\theta(\mathbf{r},t)\right)\times\psi(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t+\int_0^{t_f}\int_{A_2}\left(\lambda(\mathbf{r})\frac{\partial\theta(\mathbf{r},t)}{\partial n_2}\right)\psi(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t-\int_0^{t_f}\int_{A_3}\left(\lambda(\mathbf{r})\frac{\partial\psi(\mathbf{r},t)}{\partial n_3}\right)\theta(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t+\int_{\mathscr{R}}C(\mathbf{r})\psi(\mathbf{r},0)\theta(\mathbf{r},0)\,\mathrm{d}r+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (21)$$

Once the sensitivity function $\theta(\mathbf{r}, t)$ is determined from the solution of the sensitivity problem and the adjoint function $\psi(\mathbf{r}, t)$ obtained from the solution of the adjoint problem (16), the directional derivative $D_{\Delta Z}J(Z)$ is computed from equation (21). Utilizing its definition, given by equation (9), $D_{\Delta Z}J(Z)$ is expressed in the form of a scalar product as

$$D_{\Delta Z}J(Z) \equiv \langle\nabla J(Z),\Delta Z\rangle_E. \quad (22)$$

To illustrate the physical significance of the result given by equation (22), we examine the two examples considered previously.

Case #1. The function $f_1(\mathbf{r}, t)$ unknown. The function $f_1(\mathbf{r}, t)$ is unknown over the boundary surface A_1 and we wish to determine it by the inverse analysis. As discussed previously, $f_1(\mathbf{r}, t)$ is related to the ambient temperature $\phi(\mathbf{r}, t)$ through the relation $f_1(\mathbf{r}, t) = h(\mathbf{r})\cdot\phi(\mathbf{r}, t)$.

The sensitivity problem (13) is utilized to simplify equation (21), to obtain

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\int_{A_1}\Delta Z(\mathbf{r},t)\psi(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (23)$$

For this particular case, we have $Z \equiv f_1(\mathbf{r}, t)$ the function space $E \equiv L^2(A_1\times]0, t_f[)$. Therefore, the definition of the scalar product for this particular case is given by equation (6c). Utilizing this definition (6c), equation (23) becomes

$$\begin{aligned}D_{\Delta Z}J(Z) &= \langle\psi,\Delta Z\rangle_E+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E\\ &= \langle\psi+\alpha_0(Z-Z_{\mathrm{est}}),\Delta Z\rangle_E.\end{aligned} \quad (24)$$

Then according to equation (22), the gradient of J is a function of $(\mathbf{r}, t)$ defined on $A_1\times]0, t_f[$, given by

$$\nabla J(\mathbf{r},t;Z) = \psi(\mathbf{r},t)+\alpha_0(Z(\mathbf{r},t)-Z_{\mathrm{est}}(\mathbf{r},t)). \quad (25)$$

A special case. The function $\phi(t)$ unknown. As a further special case, we consider equation (1f), that is

$$f_1(\mathbf{r},t) = h(\mathbf{r})\phi(t) \quad (1f)$$

where $h(\mathbf{r})$ is known and $\phi(t)$ is to be determined.

We have $Z \equiv \phi(t)$, $\Delta Z = \Delta\phi(t)$ and the function space $E \equiv L^2(0, t_f)$. Therefore, the definition of the scalar product for this case is given by equation (6a). Utilizing the definition (6a), equation (23) becomes

$$\begin{aligned}D_{\Delta Z}J(Z) &= \int_0^{t_f}\Delta Z(t)\left(\int_{A_1}h(\mathbf{r})\psi(\mathbf{r},t)\,\mathrm{d}A\right)\mathrm{d}t+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E\\ &= \int_0^{t_f}\Delta Z(t)H(t)\,\mathrm{d}t+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E\\ &= \langle H,\Delta Z\rangle_E+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E\\ &= \langle H+\alpha_0(Z-Z_{\mathrm{est}}),\Delta Z\rangle_E.\end{aligned} \quad (26)$$

Thus according to the definition (22), the gradient of J is a function of t defined on $]0, t_f[$, given by

$$\nabla J(t;Z) = H(t)+\alpha_0(Z(t)-Z_{\mathrm{est}}(t)) \quad (27a)$$

where

$$H(t) = \int_{A_1}h(\mathbf{r})\psi(\mathbf{r},t)\,\mathrm{d}A. \quad (27b)$$

In the foregoing analysis, to minimize the functional $J(Z)$ in equation (4), we prefer to introduce a regularization term, i.e. $\alpha_0 > 0$, and use the stopping criteria of equation (40g). If no regularization is used (i.e. $\alpha_0 = 0$), then equation (16f) leads to $H(t_f) = 0$ and $\nabla J(t_f, Z) = 0$. For such a case, a good estimate is needed for the final time condition $Z(t_f)$. A method described in ref. [15] can be used to alleviate the inaccuracy associated with the lack of knowledge about $Z(t_f)$.

Case #2. Thermal conductivity unknown. Thermal conductivity $\lambda(\mathbf{r})$ is to be determined on $\mathcal{R}$. The sensitivity problem (15) is utilized to simplify equation (21). We obtain

$$D_{\Delta Z}J(Z) = \int_0^{t_f}\int_{\mathcal{R}} \nabla\cdot(\Delta Z(\mathbf{r})\nabla T(\mathbf{r},t))\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t$$

$$-\sum_{i=1}^{2}\int_0^{t_f}\int_{A_i}\Delta Z(\mathbf{r})\frac{\partial T(\mathbf{r},t)}{\partial n_i}\psi(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t$$

$$+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (28a)$$

Green's formula is applied to the first integral term on the right-hand side of equation (28a)

$$\int_0^{t_f}\int_{\mathcal{R}} \nabla\cdot(\Delta Z(\mathbf{r})\nabla T(\mathbf{r},t))\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t$$

$$=\sum_{i=1}^{3}\int_0^{t_f}\int_{A_i}\Delta Z(\mathbf{r})\frac{\partial T(\mathbf{r},t)}{\partial n_i}\psi(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t$$

$$-\int_0^{t_f}\int_{\mathcal{R}}\Delta Z(\mathbf{r})\nabla T(\mathbf{r},t)\nabla\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t.$$

Then equation (28a) takes the form

$$D_{\Delta Z}J(Z) \equiv -\int_0^{t_f}\int_{\mathcal{R}}\Delta Z(\mathbf{r})\nabla T(\mathbf{r},t)\nabla\psi(\mathbf{r},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}t$$

$$+\int_0^{t_f}\int_{A_3}\Delta Z(\mathbf{r})\frac{\partial T}{\partial n_3}(\mathbf{r},t)\psi(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t$$

$$+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (28b)$$

In view of the boundary condition (16e), this result simplifies to

$$D_{\Delta Z}J(Z) = -\int_{\mathcal{R}}\Delta Z(\mathbf{r})\left(\int_0^{t_f}\nabla T(\mathbf{r},t)\nabla\psi(\mathbf{r},t)\,\mathrm{d}t\right)\mathrm{d}\mathbf{r}$$

$$+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E. \quad (28c)$$

For this particular case, we have $Z\equiv\lambda(\mathbf{r})$ and the function space $E\equiv L^2(\mathcal{R})$. Therefore, the definition of the scalar product is given by equation (6b). Utilizing the definition (6b), equation (28c) becomes

$$D_{\Delta Z}J(Z) = -\int_{\mathcal{R}}\Delta Z(\mathbf{r})G(\mathbf{r})\,\mathrm{d}\mathbf{r}+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E$$

$$= -\langle G,\Delta Z\rangle_E+\alpha_0\langle Z-Z_{\mathrm{est}},\Delta Z\rangle_E$$

$$= \langle -G+\alpha_0(Z-Z_{\mathrm{est}}),\Delta Z\rangle_E. \quad (29a)$$

Thus, according to the definition (22), the gradient of J is a function of $\mathbf{r}$ defined on $\mathcal{R}$, given by

$$\nabla J(\mathbf{r};Z) = -G(\mathbf{r})+\alpha_0(Z(\mathbf{r})-Z_{\mathrm{est}}(\mathbf{r})) \quad (29b)$$

where

$$G(\mathbf{r}) = \int_0^{t_f}\nabla T(\mathbf{r},t)\nabla\psi(\mathbf{r},t)\,\mathrm{d}t. \quad (29c)$$

5. THE FINITE DIMENSIONAL CASE

The solution of the finite dimensional problem for which the unknown function Z is expressed in the form of equation (7), can be readily obtained as a special case from the generalized solution methodology presented for the infinite dimensional problem.

The only change in the solution methodology is the computation of the gradient defined by equations (21) and (22).

When Z is given by equation (7), the variation ΔZ and the gradient $\nabla J(Z)$ are given respectively by

$$\Delta Z(s) = \sum_{i=1}^{P}\Delta Z_i\sigma_i(s) \quad (30)$$

$$\nabla J(s) = \sum_{i=1}^{P}\nabla J_i\sigma_i(s). \quad (31)$$

Now we need to determine the P components ∇J_i of the gradient vector ∇J. The general procedure is similar to that described previously for the infinite dimensional case. Here we illustrate the basic approach by again considering cases #1 and #2 already presented.

Case #1. The function $f_1(\mathbf{r},t)$ unknown. The function $f_1(\mathbf{r},t)$ is unknown over the boundary surface A_1 and it is to be determined by the inverse analysis. The physical significance of $f_1(\mathbf{r},t)$ in relation to the ambient temperature is as discussed previously.

Equation (23) takes the form

$$D_{\Delta Z}J(Z) = \sum_{i=1}^{P}\Delta Z_i\gamma_i+\alpha_0\sum_{i=1}^{P}\sum_{j=1}^{P}\Delta Z_iS_{ij}(Z_j-Z_{j,\mathrm{est}}) \quad (32)$$

where

$$\gamma_i = \langle\psi,\sigma_i\rangle_E = \int_0^{t_f}\int_A\psi(\mathbf{r},t)\sigma_i(\mathbf{r},t)\,\mathrm{d}A\,\mathrm{d}t \quad (33)$$

$$S_{ij} = \langle\sigma_i,\sigma_j\rangle_E,\quad i=1,\ldots,P;\ j=1,\ldots,P. \quad (34)$$

Equation (32) can be rearranged as

$$D_{\Delta Z}J(Z) = \sum_{i=1}^{P}\Delta Z_i\nabla J_i \quad (35)$$

where

$$\nabla J_i = \gamma_i+\alpha_0\sum_{j=1}^{P}S_{ij}(Z_j-Z_{j,\mathrm{est}}),\quad i=1,\ldots,P. \quad (36)$$

Thus, once $\psi(\mathbf{r},t)$ is available from the adjoint problem, ψ_i is determined from equation (33) and the P components of the gradient ∇J_i are determined from equation (36) since the basis functions σ_i are known.

Case #2. The thermal conductivity unknown. Equation (28c) takes the form

$$D_{\Delta Z}J(Z) = \sum_{i=1}^{P} \Delta Z_i \gamma_i^* + \alpha_0 \sum_{i=1}^{P} \sum_{j=1}^{P} \Delta Z_i S_{ij}(Z_j - Z_{j,\text{est}}) \quad (37)$$

where

$$\gamma_i^* = \langle -G, \sigma_i \rangle_E = \int_{\mathscr{R}} -G(\mathbf{r})\sigma_i(\mathbf{r})\, d\mathbf{r} \quad (38)$$

$G(\mathbf{r})$ is the same as equation (29c) and S_{ij} is the same as equation (34). Hence, the P components ∇J_i of the gradient are determined from

$$\nabla J_i = \gamma_i^* + \alpha_0 \sum_{j=1}^{P} S_{ij}(Z_j - Z_{j,\text{est}}), \quad i = 1, \ldots, P. \quad (39)$$

6. THE SOLUTION ALGORITHM

The inverse problem has been formulated in Section 2 as an infinite dimensional optimization problem of the functional J defined by equation (4). Now, we need to find Z such that it will minimize $J(Z)$, i.e.

$$\min_{Z \in E} J(Z).$$

Having the sensitivity and adjoint problems and the gradient function $\nabla J(Z)$ available, we now describe the *conjugate gradient method* for minimizing the functional $J(Z)$ in order to solve the inverse problem.

6.1. *Conjugate gradient method*

The basic steps in the application of the Polak–Ribiere version [16] of the conjugate gradient method applied to the solution of the above problems is described below.

Step #1. Choose an initial guess $Z^0 \in E$; for example, $Z^0 = Z_{\text{est}}$, or $Z^0 = \text{constant}$. Set $n = 0$.

Step #2. Define the scalar β^n:

$$\beta^n = 0, \quad \text{if } n = 0 \quad (40a)$$

$$\beta^n = \frac{\langle \nabla J(Z^n), \nabla J(Z^n) - \nabla J(Z^{n-1}) \rangle_E}{\|\nabla J(Z^{n-1})\|_E^2}, \quad \text{if } n > 0. \quad (40b)$$

Step #3. Define the direction $p^n \in E$:

$$p^0 = \nabla J(Z^0), \quad \text{if } n = 0 \quad (40c)$$

$$p^n = \nabla J(Z^n) + \beta^n p^{n-1}, \quad \text{if } n > 0. \quad (40d)$$

Step #4. Define the step size $\rho^n \in \mathscr{R}$

$$\rho^n = \text{Arg min } J(Z^n - \rho p^n). \quad (40e)$$

Step #5. Set

$$Z^{n+1} = Z^n - \rho^n p^2. \quad (40f)$$

Step #6. If

$$\|Z^{n+1} - Z^n\|_E^2 < \varepsilon, \text{ stop.} \quad (40g)$$

Otherwise, set $n = n+1$, go to step #2.

6.2. *Implementation of the conjugate gradient method for solving inverse problem*

Depending on the nature of the unknown function Z to be determined, the different steps of the above algorithm can be implemented in the following way:

Steps #2 and #3:

- solve the direct problem (1), forward in time, to obtain the temperature $T(d_m, t; Z^n)$;
- compute the error terms $e_m(t; Z^n)$ from equation (16a) and the functional $J(Z^n)$ from equation (4);
- solve the adjoint problem (16b)–(16f), backward in time, to obtain $\psi(\mathbf{r}, t; Z^n)$;
- compute the gradient $\nabla J(Z^n)$ from the gradient equation (21);
- compute the scalar β^n and the direction p^n from equations (40a) to (40c).

Remark. Depending on the nature of the problem, the general equation (21) needs to be simplified and expressed in terms of the adjoint function ψ before computing the gradient $\nabla J(Z^n)$. For example, for the problem of determining $f_1(\mathbf{r}, t)$, equation (21) reduces to equation (25) and for the thermal conductivity problem, it reduces to equation (29).

Step #4: *in the computation of the step size* ρ^n, two different cases need to be distinguished:

(i) sensitivity function $\theta(\mathbf{r}, t)$ depends on temperature $T(\mathbf{r}, t)$;
(ii) sensitivity function $\theta(\mathbf{r}, t)$ is independent of $T(\mathbf{r}, t)$.

This matter is illustrated below by examining the two cases considered previously.

Case #1. The function $f_1(\mathbf{r}, t)$ unknown. The function $f_1(\mathbf{r}, t)$ is to be determined on the boundary surface A_1. Because of the linearity of the direct problem (1), the functional $J(Z)$ is quadratic with respect to Z as shown below.

The linearity of $T(Z)$ in Z implies that

$$T(\mathbf{r}, t; Z - \rho\Delta Z) = T(\mathbf{r}, t; Z) - \rho D_{\Delta Z} T(\mathbf{r}, t; \Delta Z) = T(\mathbf{r}, t; Z) - \rho\theta(\mathbf{r}, t) \quad (41)$$

where $\theta(\mathbf{r}, t)$ is the solution of the sensitivity problem (13), and ρ is a scalar.

Let us set $\Delta Z(\mathbf{r}, t) = p^n(\mathbf{r}, t)$ where p^n is defined by equation (40d) and set $\Delta f_1(\mathbf{r}, t) = \Delta Z(\mathbf{r}, t) = p^n(\mathbf{r}, t)$ in the sensitivity problem (13). The resulting solution of problem (13) is denoted by $\theta^n(\mathbf{r}, t)$. Then the functional J defined by equation (4), at the point $(Z^n - \rho p^n)$ takes the form

$$J(Z^n - \rho p^n) = \frac{1}{2} \int_0^{t_f} \sum_{m=1}^{M} |T(d_m, t; Z^n - \rho p^n) - \bar{Y}_m(t)|^2 \, dt + \alpha_0/2 \|Z^n - \rho p^n - Z_{\text{est}}\|_E^2 \quad (42a)$$

$$J(Z^n-\rho p^n)=\frac{1}{2}\int_0^{t_f}\sum_{m=1}^{M}|T(d_m,t;Z^n)-\rho\theta^n(d_m,t)$$

$$-\bar{Y}_m(t)|^2\,dt+\alpha_0/2\|Z^n-\rho p^n-Z_{est}\|_E^2 \quad (42b)$$

$$J(Z^n-\rho p^n)=\frac{1}{2}\left(\int_0^{t_f}\sum_{m=1}^{M}|T(d_m,t;Z^n)-\bar{Y}_m(t)|^2\,dt\right.$$

$$\left.+\alpha_0\|Z^n-Z_{est}\|_E^2\right)-\rho\left(\int_0^{t_f}\sum_{m=1}^{M}(T(d_m,t;Z^n)\right.$$

$$\left.-\bar{Y}_m(t))\theta^n(d_m,t)\,dt+\alpha_0\langle Z^n-Z_{est},p^n\rangle_E\right)$$

$$+\frac{\rho^2}{2}\left(\int_0^{t_f}\sum_{m=1}^{M}|\theta^n(d_m,t)|^2\,dt+\alpha\|p^n\|_E^2\right). \quad (42c)$$

Utilizing the definition of the gradient $\nabla J(Z)$ given by equation (9b) and the directional derivative expressed in the form given by equation (17), we obtain

$$J(Z^n-\rho p^n)=J(Z^n)-\rho\langle\nabla J(Z^n),p^n\rangle_E+\frac{\rho^2}{2}K^n \quad (43a)$$

where K^n is the constant given by

$$K^n=\int_0^{t_f}\sum_{m=1}^{M}|\theta^n(d_m,t)|^2\,dt+\alpha_0\|p^n\|_E^2. \quad (43b)$$

Then according to equation (43a), $J(Z^n-\rho p^n)$ is quadratic in ρ. This is a general result which is valid when the solution of the direct problem $T(Z)$ is linear with respect to the function Z to be determined.

Hence the best value ρ^n which minimizes $J(Z^n-\rho p^n)$ according to equation (40e) is obtained by minimizing equation (43a) with respect to ρ

$$\rho^n=\frac{\langle\nabla J(Z^n),p^n\rangle_E}{K^n}. \quad (44)$$

The sensitivity function $\Theta^n(\mathbf{r},t)$ needed in the computation of equation (44) is determined from the solution of the sensitivity problem by setting $\Delta Z\equiv p^n$. The solution algorithm is shown schematically in Table 1.

Case #2. The thermal conductivity unknown. The thermal conductivity $\lambda(\mathbf{r})$ is to be determined. In that case the solution $T(Z)$ of the direct problem (1) is not linear with respect to the unknown function $Z\equiv\lambda$. Then $J(Z)$ is not quadratic as in the previous case. So the determination of the scalar ρ^n may be achieved by dichotomy or Fibonacci methods [16] in order to satisfy equation (40e).

Another approach consists of linearization of $T(\mathbf{r},t;Z+\rho\Delta Z)$, that is, by considering the approximation

$$T(\mathbf{r},t;Z-\rho\Delta Z)\cong T(\mathbf{r},t;Z)-\rho D_{\Delta Z}T(\mathbf{r},t;\Delta Z) \quad (45)$$

$$\cong T(\mathbf{r},t;Z)-\rho\theta(\mathbf{r},t). \quad (46)$$

For such a case, the solution methodology described previously becomes applicable.

Table 1. Solution algorithm for the conjugate gradient method

Steps	Computations	Equations
1	Z^n available	
2	the error terms $e_m(t;Z^n)$	(16a)
	the functional $J(Z^n)$	(4)
	the adjoint function $\psi(\mathbf{r},t;Z^n)$	(16b)–(16f)
	the gradient $\nabla J(Z^n)$	(25), (27), (29), (36) or (39)
	the scalar β^n	(40a), (40b)
3	the direction p^n	(40c), (40d)
4	the sensitivity function $\theta(\mathbf{r},t)$	(13), (15), etc.
	the scalar ρ^n	(40e), (43b), (44)
5	Z^{n+1}	(40f)
6	apply the stopping criteria	(40g)

7. CONCLUSIONS

A general methodology has been proposed for formulating the solution of the inverse heat conduction problem (IHCP) as the solution of infinite dimensional optimization problem (IDOP). The approach assumes no a priori information on the nature of the unknown functions to be determined by the inverse analysis. Finite dimensional problems, which occur when a priori information is available, become merely a special case of the present approach.

A general algorithm has been presented for solving IHCP by iteration. When the direct problem is linear with the unknown function to be determined, then the functional to be minimized is quadratic convex, the solution is unique and the convergence of the sequence defined by the conjugate gradient method is guaranteed if some regularization is introduced, i.e. if the regularization parameter α_0 is positive. In the case the direct problem is nonlinear with the unknown function, the functional to be minimized may have local minima. The method allows the determination of such minima by varying the initial guess.

Illustrative examples have been presented for the determination of unknown surface heat flux, ambient temperature and thermal conductivity. The method applies to the determination of other properties such as heat capacity, surface temperature, spatially varying heat transfer coefficient, initial temperature, generation term. The steady-state inverse heat conduction problems are also a special case of the present method.

Acknowledgement—One of the authors (M. N. Ozisik) would like to acknowledge the hospitality of I.S.I.T.E.M. during his visit with the University of Nantes and the support through the NSF grant MSS 88-16107.

REFERENCES

1. J. V. Beck, B. Blackwell and C. R. St. Clair, Jr., *Inverse Heat Conduction*. Wiley, New York (1985).
2. A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-posed Problems*. Winston, Washington, DC (1977).

3. P. Carré and D. Delaunay, Simultaneous measurement of the thermal properties of phase change material and complex liquids using a nonlinear thermocinetic model, XVth Int. Symp. of Heat and Mass Transfer, Dubrovnik (1983).
4. G. P. Flach and M. N. Ozisik, Inverse heat conduction problem of periodically contacting surfaces, *J. Heat Transfer* **110**, 821–829 (1988).
5. B. Leden, M. H. Hamza and M. A. Shiera, Different methods for estimation of thermal diffusivity of a heat diffusion process, 3rd IFAC Symp., Identification and System Parameter Estimation, The Hague/Delft, The Netherlands (1973).
6. M. P. Polis, R. E. Goodson and M. H. Wozny, On parameter identification for distributed systems using Galerkin's criterion, *Automatica* **9**, 53–64 (1983).
7. J. V. Beck, Sequential estimation of thermal parameters, *J. Heat Transfer* **99**(2), (1977).
8. G. Chavent, Identification of distributed parameter systems about the output least square and its implementation and identifiability, 5th IFAC Symp., Identification and Systems Parameter Estimation, Darmstadt, F.R.G. (1979).
9. O. M. Alifanov and S. V. Rumyantsev, One method of solving incorrectly stated problems, *J. Engng Phys.* **34**(2), 328–331 (1978).
10. S. V. Rumyantsev, Ways of allowing for a priori information in regularizing gradient algorithms, *J. Engng Phys.* **49**(3), 932–936 (1986).
11. O. M. Alifanov and S. V. Rumyantsev, Application of iterative regularization for the solution of incorrect inverse problems, *J. Engng Phys.* **53**(5), 1335–1342 (1987).
12. Y. Jarny, D. Delaunay and J. Bransiet, Identification of nonlinear thermal properties by an output least square method, *Proc. Int. Heat Transfer Conf.*, pp. 1811–1816 (1986).
13. M. El Bagdouri and Y. Jarny, Optimal boundary control of a thermal system: an inverse conduction problem, 4th IFAC Symp. Control of Distributed Parameter Systems, Los Angeles, California (1986).
14. P. K. Lamm, Regularization and the adjoint method of solving inverse problems. Lectures given at 3rd Annual Inverse Problems in Engng Seminar, Michigan State University, East Lansing, U.S.A., 25–26 June (1990).
15. O. M. Alifanov and Y. V. Egerov, Algorithm and results of solving inverse heat-conduction boundary problems in a two-dimensional formulation, *J. Engng Phys.* **48**, 489–496 (1985).
16. E. Polak, *Computational Methods in Optimization.* Academic Press, New York (1971).

UNE METHODE GENERALE D'OPTIMISATION POUR LA RESOLUTION DE PROBLEMES INVERSES MULTIDIMENSIONNELS DE CONDUCTION

Résumé—Une formulation est présentée pour résoudre des problèmes inverses de conduction 3D comme un problème général d'optimisation, par un algorithme de gradient conjugué. Cette formulation comporte les equations de sensibilité, les équations adjointes et les équations du gradient. Un algorithme est décrit pour obtenir les conditions surfaciques (flux de chaleur ou température), les variations spatiales de la conductivité et de la capacité thermique, à partir d'enregistrements de la température, effectués au sein du solide. Dans cette approche du problème, aucun choix a priori est nécessaire sur la fonction inconnue à déterminer. On montre que les problèmes comportant un choix a priori de la fonction inconnue deviennent des cas particuliers de cette approche générale.

EINE ALLGEMEINGÜLTIGE OPTIMIERUNGSMETHODE ZUR LÖSUNG MEHRDIMENSIONALER PROBLEME DER INVERSEN WÄRMELEITUNG UNTER VERWENDUNG DES KONZEPTES DER ADJUNGIERTEN GLEICHUNG

Zusammenfassung—Es wird ein Ansatz zur Lösung dreidimensionaler Probleme der inversen Wärmeleitung vorgestellt. Zur Behandlung des verallgemeinerten Optimierungsproblems wird das Konzept der adjungierten Gleichung in Verbindung mit dem konjugierten Gradientenverfahren benutzt. Die Formulierung besteht aus dem Empfindlichkeitsproblem, dem adjungierten Problem und den Gradientengleichungen. Es wird ein Lösungsalgorithmus zur Bestimmung der Oberflächenbedingungen (Wärmestromdichte oder Temperatur), der örtlichen Wärmeleitfähigkeit und der Wärmekapazität vorgestellt. Der Algorithmus basiert auf den aufgezeichneten zeitlichen Temperaturverläufen innerhalb des Festkörpers. Bei diesem Ansatz wird keine Vorabinformation über die zu ermittelnde Funktion benötigt. Liegen bereits Informationen über die Funktion vorab fest, so stellt dies einen Sonderfall des allgemeinen Lösungsweges dar.

ОБЩИЙ МЕТОД ОПТИМИЗАЦИИ С ИСПОЛЬЗОВАНИЕМ СОПРЯЖЕННОГО УРАВНЕНИЯ ДЛЯ РЕШЕНИЯ ЗАДАЧИ МНОГОМЕРНОЙ ОБРАТНОЙ ТЕПЛОПРОВОДНОСТИ

Аннотация—Представлена трехмерная формулировка обратной задачи теплопроводности как общей задачи оптимизации с использованием метода сопряженных уравнений в сочетании с алгоритмом градиента. Формулировка включает задачу устойчивости, сопряженную задачу и уравнения градиента. Приводится алгоритм решения для оценки условия на поверхности (т.е. теплового потока или температуры), а также зависящих от координат коэффициента теплопроводности и теплосодержания на основе данных по нестационарной температуре, регистрируемой внутри твердого тела. При используемом подходе отсутствует необходимость в априорной информации об искомой функции. Показано, что задачи, в которых используется такая информация, представляют особые случаи.